Chapter 1

1.1

- Microscopic: $\{p_i, q_i\}$, $\{s_i\}$, $\{n_i\}$ etc

$\downarrow$ Probability

- Macroscopic: P, V, T, E (AKA U), S

Probability: $P(\mu) = \exp[-\beta\mathcal{H}(\mu)]/Z$

$$F = -\frac{1}{\beta}\log Z$$

This book: Mesoscopic: Φ, $M = \langle s \rangle$

→ Classify phases of matter with this

1.2 Phonons

$$V(\underbrace{q_1 \cdots q_N}_{\text{ionic coords}})$$

Minima at $q^*_{lmn} = l\hat{a} + m\hat{b} + n\hat{c}$

$$V = V^* + \sum_{\substack{r,r'\\ \alpha,\beta}} \frac{\partial^2 V}{\partial q_{r\alpha}\partial q_{r'\beta}} u_\alpha(r) u_\beta(r') + O(u^3)$$

$$\mathcal{H} = \sum_{r,\alpha} \frac{p_\alpha(r)^2}{2m} + V$$

$$\frac{\partial^2 V}{\partial q_{r\alpha}\partial q_{r'\beta}} = K_{\alpha\beta}(r-r')$$

$$\Rightarrow u_\alpha(r) = \sum_k \frac{e^{ikr}}{\sqrt{N}} u_\alpha(k)$$

$$\Rightarrow \mathcal{H} = V^* + \sum_{\vec{k}} \frac{|p(k)|^2}{2m} + u(\vec{k})\cdot K(k)\cdot u(\vec{k})$$

$\downarrow$ diag in evecs

$$= V^* + \sum_{k,\alpha}\left[\frac{(p_\alpha(k))^2}{2m} + \chi_\alpha(k)\,\tilde{u}_\alpha(\vec{k})\,\tilde{u}_\alpha(\vec{k})\right]$$

$$= V^* + \sum_{\vec{k},\alpha} \hbar\,\omega_\alpha(\vec{k})\left(n_\alpha(\vec{k}) + \tfrac{1}{2}\right) \qquad \omega_\alpha(k) = \sqrt{\frac{\chi_\alpha(k)}{m}}$$

$$\langle n_\alpha(k)\rangle = \frac{1}{e^{\beta\omega_\alpha\hbar} - 1}$$

$\chi_\alpha(k)$ gives rise to nontrivial behavior

1D simplification:

$$V = V^* + \frac{K_1}{2}\sum_n (u_{n+1} - u_n)^2 + \frac{K_2}{2}\sum_n (u_{n+2} - u_n)^2 + \cdots$$

$$u_n = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi}\, e^{-ikna}\, u(k) \qquad u(k) = \sum_n u_n\, e^{ikna}$$

Brillouin zone

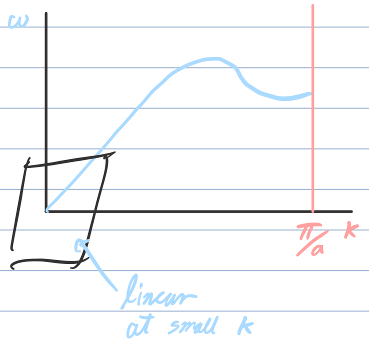
$$\Rightarrow V = V^* + \frac{K_1}{2}\sum_n \int dk_1 dk_2\, (e^{ik_1 a} - 1)(e^{ik_2 a} - 1)\, \underbrace{e^{-i(k_1 - k_2)an}}_{\delta(k_1 + k_2)\frac{2\pi}{a}}\, u(k_1)\, u(k_2) + \cdots$$

$$= V^* + \int \frac{dk}{a}\, [\underbrace{K_1(1 - \cos ka) + K_2(1 - \cos 2ka) + \cdots}_{m\omega^2}]\, |u(k)|^2$$

$$\omega(k) = \sqrt{\frac{2K_1(1 - \cos ka) + \cdots}{m}}$$

as $k \to 0$ $\quad \omega(k) \approx k \cdot v \quad v = a\sqrt{\frac{\bar{K}}{m}} \quad \bar{K} = \sum n^2 K_n$

higher K_n's change v but not the $E \sim T^2$ scaling



T-indep

$$E(T) = V^* + Na\int dk\, \hbar\omega(k)\left[\frac{1}{\exp\frac{\hbar\omega(k)}{k_B T} - 1} + \frac{1}{2}\right]$$

$T \to 0 \Rightarrow$ only smallest $\omega(k)$ matters

$$E(T) = \tilde{V}^* + Na\int dk \frac{\hbar v|k|}{\exp\frac{\hbar v|k|}{k_BT} - 1} = \tilde{V}^* + \frac{Na}{\hbar v}\frac{\pi^2}{6}(k_BT)^2$$

$$\Rightarrow C = \frac{dE}{dT} \sim T \leftarrow \text{universal!}$$

Field Approach (Phenomenological)

"Mesoscopic"

$\lambda_{\text{typical}} > \lambda(T) \approx \frac{\hbar v}{k_BT} \gg a$

$u(x)$ is then the "avg displacement" and varies slowly over dx

$a \ll dx \ll \lambda(T)$

$\sum_n \to \frac{1}{a}\int dx$

$\dot{u} = \frac{\partial u}{\partial t} \Rightarrow$ Kinetic term $= \underbrace{\frac{m}{a}}_{\rho}\int dx \frac{(\dot{u})^2}{2}$

$V[u]$ not generally known but by

1) Locality $\Rightarrow V[u] = \int dx\, \mathcal{L}(u, \partial_x u, \partial_x^2 u, \dots)$
2) Trans. Symm (no explicit x-dep or even $u(x)$-dep)
3) Stability (no linear term in u or ∂u, highest order term is even w coeff > 0)

$$\Rightarrow V[u] = \int dx \left[\frac{K}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{L}{2}\left(\frac{\partial^2 u}{\partial x}\right)^2 + \mu\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x} + \dots\right]$$

$$= \int dk \left[\frac{K}{2}k^2 + \frac{L}{2}k^4\right]|u(k)|^2 - i\mu\int dk_1\, dk_2\; k_1 k_2 (k_1+k_2)^2 u(k_1)u(k_2)u(-k_1-k_2) + \dots$$

as $k \to 0$ only first term matters

$$\Rightarrow \mathcal{L} = \frac{\rho}{2}\int dx \left[\left(\frac{\partial u}{\partial t}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2\right] \qquad v = \sqrt{\frac{K}{\rho}}$$

$$= \frac{\rho}{2}\int dk \left[\omega^2 + v^2k^2\right]|u|^2$$

$$\Rightarrow \omega = v|k|$$

In general dimensions

Most general $\mathcal{H}$ in terms of irreps at second order

$u \rightarrow u_\alpha(\vec{x})$

(Einsum)

$$\mathcal{H} = \frac{1}{2}\int d^d x \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 + 2\mu\, u_{\alpha\beta} u_{\alpha\beta} + \lambda u_{\alpha\alpha} u_{\beta\beta}\right]$$

$u_{\alpha\beta} = \partial_\alpha u_\beta)$

$$= \frac{1}{2}\int d^d x \left[\rho|\dot{u}|^2 + \mu k^2|u|^2 + (\mu+\lambda)|k\cdot u|^2\right]$$

$v_\ell = \sqrt{(2\mu+\lambda)/\rho}$ $\quad u \parallel k$

$v_t = \sqrt{\mu/\rho}$ $\quad u \perp k$

$$\Rightarrow E(T) = L^d \int d^d k \; \frac{\hbar v_\ell k}{\exp\frac{\hbar v_\ell k}{k_B T} - 1} + \frac{\hbar v_t k}{\exp\frac{\hbar v_t k}{k_B T} - 1}$$

$$\approx A(v_\ell, v_t)\, L^d (k_B T)^{d+1}$$

$\Rightarrow C \sim T^d$ as $d \rightarrow 0$

In superfluid helium $C \sim T^3$

$C \sim T^{3/2}$ for ideal bose gas

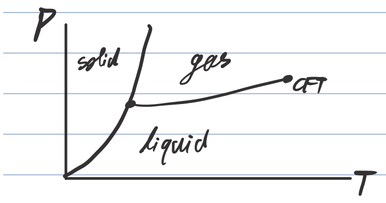
Diffusion: $x \propto \sqrt{Dt}$

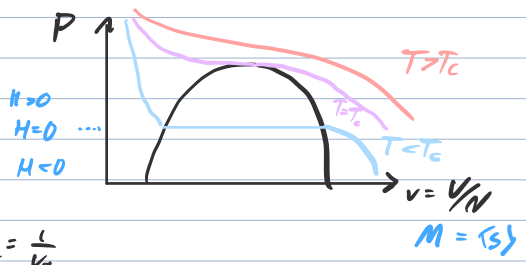
Transport: $x \propto vt$

Free forced : $x \propto gt^2/2$

Unlike the example before, generally cant ignore nonlin terms

1.3 Phase Transitions





$\rho_\ell = \frac{1}{v_\ell} \quad \rho_g = \frac{1}{v_g}$

$$\rho_\ell - \rho_g \sim \frac{v_g - v_\ell}{v_g v_\ell}$$

$$\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial p}\Big|_T \quad \Rightarrow \quad \chi \to \infty \quad \text{as} \quad T \to T_c$$

1.4 Critical Behavior

total magnetization

Order param: $\frac{1}{V}\lim_{h\to 0^+} \overset{\swarrow}{M}(h,T) =: m(T)$

$$m(T, h=0) \propto \begin{cases} 0 & T>T_c \\ |t|^{-\beta} & T<T_c \end{cases}$$

$$t = \frac{T-T_c}{T_c}$$

$$m(T_c, h) \propto h^{1/\delta}$$

Response Function: $\chi_\pm(T, h=0) = |t|^{-\gamma_\pm}$

$$\chi = \frac{\partial m}{\partial h}$$

Usually $\gamma_+ = \gamma_-$

$\alpha_+ = \alpha_-$

$$C_\pm(T, h=0) = |t|^{-\alpha_\pm}$$

$$c = \frac{\partial u}{\partial T}$$

Long-Range Correlations:

$$Z(h) = \mathrm{Tr}\,\exp(-\beta\mathcal{H} + \beta h M)$$

$$\frac{\partial \log Z}{\partial \beta h} = \langle M\rangle \qquad M = \int d^3r\; m(r)$$

$$\Rightarrow \quad \chi = \frac{\partial M}{\partial h} = \beta\langle M^2\rangle_c$$

$$\Rightarrow \quad k_B T\chi = V\cdot\int d^3x\; \underbrace{\langle m(r)\, m(0)\rangle_c}_{G_c(r) \sim \exp[-r/\xi]}$$

For $r > \xi$

$$\Rightarrow \frac{k_B T \chi}{V} \sim g \xi^3$$

$\chi \to \infty$ as $T \to T_c$

$\Rightarrow \xi \to \infty$

$\nu_+ = \nu_-$

$$\xi(T, 0) = |t|^{-\nu_\xi}$$

2 Statistical Fields

2.1 Intro:

In full generality

$$Z = \mathrm{Tr} \exp -\beta \mathcal{H}_{mic}$$

but long-wavelengths matter more!

$\Rightarrow$ define $\vec{m}(x)$ as an average over $d^d x > a^d$

$\Rightarrow$ no variation / Fourier modes die beyon $k = \Lambda \sim 1/a$

☆☆

$Z[T]$ is preserved!!!

$$Z[T] = \mathrm{Tr} \exp -\beta \mathcal{H}_{mic} = \int D\vec{m} \; \underset{\substack{\uparrow \\ \text{pushed-forward probability}}}{W[\vec{m}(x)]}$$

$$m: \mathbb{R}^d \to \mathbb{R}^n$$

$d=4$ $n=1$ $\Rightarrow$ scalar QFT etc

$$\mathcal{H}(m(x)) := -\frac{1}{\beta} \log W[\vec{m}(x)]$$

Locality & Uniformity:

$$\Rightarrow \beta \mathcal{H} = \int d^d x \; \Phi[x, \vec{m}(x), \nabla m, \nabla^2 m]$$

uniform $\Rightarrow$ no x-dependence explicitly

For sufficiently short-range interactions, only need low-order derivatives

Analyticity:

We want to expand Φ in powers of m & its derivatives

Because of the central limit theorem, we expect non-analyticities of microscopic degrees of freedom wash out.

The non analyticities in βF come because $N \to \infty$ not from $a \to 0$

Symmetries:

e.g. $\mathcal{H}[R_n \vec{m}] = \mathcal{H}[\vec{m}]$

$\Rightarrow$ linear term doesn't work

$m^2 = \vec{m} \cdot \vec{m}$ works

$m^4 = (\vec{m} \cdot \vec{m})^2$ $\quad m^6 = (\vec{m} \cdot \vec{m})^3$

$|\nabla m|^2 := \partial_\alpha m_i \, \partial_\alpha m_i$ $\qquad |\partial_1 m|^2 + \alpha |\partial_2 m|^2$

↑ isotropic in spatial direction

↑ can be removed after rescaling x_1, x_2

$\nabla^2 m = \sum_i (\nabla^2 m_i)^2$ ← isotropic

$m^2 (\nabla m)^2$ ← isotropic

but there are higher order (quartic) terms that are aniso & cannot be rescaled out

2.2 Landau - Ginzburg Hamiltonian

$$\beta \mathcal{H} = \beta F_0 + \int d^d x \left[\frac{t}{2} m^2 + u\, m^4 + \frac{K}{2} (\nabla m)^2 + \cdots - \vec{h} \cdot \vec{m} \right]$$

$u > 0$ by stability

$\vec{h} = \beta B$

Note t, u, h etc are not directly interpretable
in terms of T, T_c etc
but they are analytic in T etc

2.3 Saddle point approx

$$Z = \int D\vec{m}_{(x)} \exp[-\beta \mathcal{H}[m(x)]]$$

note $$\int D\vec{m}(x)\, \mathcal{F}[\vec{m}(x), \nabla m, \ldots] = \lim_{N \to \infty} \prod_{i=1}^{N} d\vec{m}_i\, \mathcal{F}\left[\vec{m}_i, \frac{\vec{m}_{i+1} - \vec{m}_i}{a}\right]$$

Find MLE for $\vec{m}$
1) $\nabla m = 0 \Rightarrow m$ uniform

$$\Rightarrow Z \approx Z_{sp} = e^{-\beta F_0} \int d\vec{m} \exp\left[-V\left(\frac{t}{2} m^2 + u m^4 + \ldots - \vec{h} \cdot \vec{m}\right)\right]$$

$$\beta F_{sp} = -\log Z_{sp} = \beta F_0 + V \min \Psi(\vec{m})$$

$$\Psi = \frac{t}{2} m^2 + u (m^2)^2 + \cdots - \vec{h} \cdot \vec{m}$$

$$\Psi'(m) = t\bar{m} + 4u\, \bar{m}^3 - h \quad = 0$$

$t < 0 \Rightarrow m = (-t/4u)^{1/2}$ $\quad \beta = 1/2$

$t > 0 \Rightarrow m \approx h/t$ $\quad \gamma = 1$

$t = 0 \Rightarrow m \approx \left(\frac{h}{4u}\right)^{1/3}$ $\quad \delta = 3$

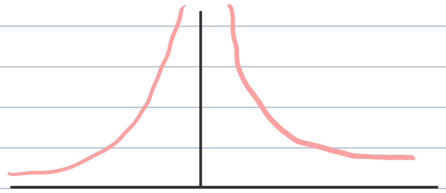
we only get these after observing

$t = a_0 + a_1 (T - T_c) + O((T - T_c)^2)$

$a_0 = 0 \quad a_1 > 0$

$$\chi_\ell^{-1} = \left.\frac{\partial h}{\partial m}\right|_{h=0} = t + 12 u \bar{m}^2 = \begin{cases} t & t > 0 \\ -2t & t < 0 \end{cases}$$

$\chi_\pm \sim A_\pm |t|^{-\gamma_\pm}$ $\qquad \frac{A_+}{A_-} = 2$ ← also universal



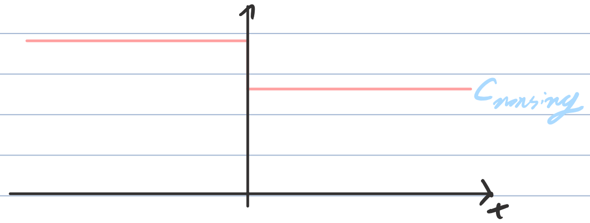
depends on t in a nonsingular way

$$\alpha: \quad \beta F = \beta F_0 + V\Psi(\bar m) = \underset{\downarrow}{\beta F_0} + V\begin{cases} 0 & t>0 \\ -\frac{t^2}{16u} & t<0 \end{cases}$$

$$C = -T\frac{\partial^2 F}{\partial T^2} \approx -T_c a^2 \frac{\partial^2}{\partial t^2}(k_B T_c \beta F) = \underset{\uparrow}{C_0} + V k_B T_c^2 a^2 \begin{cases} 0 \\ -\frac{1}{8u} \end{cases}$$

nonsing

⇒ $\alpha = 0$



2.4 Goldstone Modes

say $\mathcal{H}[\vec m] = \mathcal{H}[R\vec m]$

then $\vec m(x) \to R(x)\vec m(x)$ w/ $R(x)$ slowly varying
costs very little

E.g. Superfluid

$$\psi(x) := \psi_1(x) + i\psi_2(x) = |\psi| e^{i\theta}$$

θ should not appear physically

$$\Rightarrow \beta H = \beta F_0 + \int d^dx \left[\frac{K}{2}(\nabla\psi)^2 + \frac{t}{2}|\psi|^2 + u|\psi|^4 + \dots\right]$$

$n=2$ Landau-Ginzburg

Consider now $\Psi = \bar{\Psi} e^{i\theta(x)}$

$$\Rightarrow \beta \mathcal{H} = \underset{\mathcal{H}[\bar{\Psi}]}{\underset{\uparrow}{\beta \mathcal{H}_0}} + \frac{\bar{K}}{2} \int d^d x \, (\nabla \theta)^2$$

$\uparrow$ $\bar{K} = K \bar{\Psi}^2 \Rightarrow \bar{K} \propto \bar{\Psi}^2$

"stiffness for θ"

$$\theta(x) = \frac{1}{\sqrt{V}} \sum_q e^{iq\cdot x} \theta_q \Rightarrow \beta \mathcal{H} = \beta \mathcal{H}_0 + \frac{\bar{K}}{2} \sum_q q^2 |\theta_q|^2$$

$\Rightarrow$ energy of goldstone mode $\propto q^2$

(very small as $\lambda \to \infty$)

2.5 Domain walls

For $n=1$ $m(x \to -\infty) = -\bar{m}$
$m(x \to \infty) = m$

EOM $$\frac{d^2 m}{dx^2} = tm + 4um^3$$

$$\Rightarrow m = \bar{m} \tanh\left[\frac{x - x_0}{w}\right]$$

$$w = \sqrt{\frac{2K}{-t}} \qquad \bar{m} = \sqrt{\frac{-t}{4u}}$$

as $t \to 0$ $w \to \infty$ as $t^{-1/2}$

it turns out $w \propto \xi$ goes as $t^{-1/2}$

$$\beta F_w = \beta F[m_w] - \beta F[\bar{m}]$$

$$= \frac{2}{3}(-t)\,\bar{m}^2\, w A$$

cost for a wall to be made

$\sim t$ $\quad \sim t^{-1/2}$

$\propto (-t)^{3/2}$

cross-sectional area

3 Fluctuations

$$k_i \rightarrow \square \rightarrow k_i + q = k_s$$

sample

$|k_i| = |k_s| =: k$ for elastic

$$|q|^2 = |k_s - k_i|^2 = k^2(2 - 2\cos\theta)$$
$$= 4k^2 \sin^2 \theta/2$$
$$\Rightarrow |q| = 2k \sin \theta/2$$

Fermi's Golden rule:

$$A(q) \propto \langle k_s^f | U | k_i^i \rangle \propto \sigma(q) \underbrace{\int d^dx\, e^{iq\cdot x} \rho(x)}_{\rho(q)}$$

local form factor

what we care about

$$S(q) \propto \langle |A(q)|^2 \rangle_{\text{temporal}} \approx \langle |A(q)| \rangle_{\text{thermal}} \propto \langle |\rho(q)| \rangle_{\text{thermal}}$$

Observed scattering intensity

Ergodicity

Uniform density $\Rightarrow \rho(q) = \delta(q=0) \Rightarrow$ only fwd

Long-wavelength fluctuations $\Rightarrow$ small θ or small k probes

By Landau-Ginzburg:

$$P[\vec{m}(x)] \propto \exp\left[-\int d^dx\left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + u m^4\right]\right]$$

MLE: $\vec{m}(x) = \bar{m}\,\hat{e}_1$

MLE + fluctuations: $\vec{m}(x) = (\bar{m} + \varphi_\ell(x))\,\hat{e}_1 + \sum_{\alpha=2}^{n} \varphi_{t,\alpha}(x)\,\hat{e}_\alpha$

$$\Rightarrow (\nabla m)^2 = (\nabla\varphi_\ell)^2 + |\nabla\varphi_t|^2$$

$$m^2 = \bar{m}^2 + 2\bar{m}\varphi_\ell + \varphi_\ell^2 + |\varphi_t|^2$$

$$m^4 = \bar{m}^4 + 4\bar{m}^3\varphi_\ell + 6\bar{m}^2\varphi_\ell^2 + 2\bar{m}^2|\varphi_t|^2 + O(\varphi_t^3, \varphi_\ell^3)$$

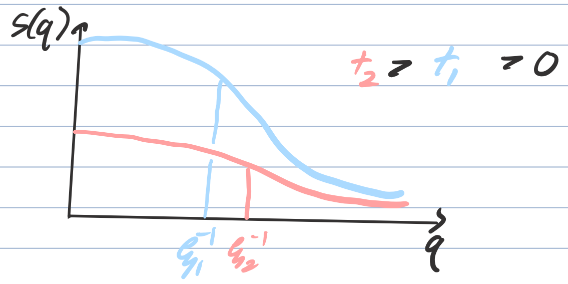
$$\beta\mathcal{H} = -\log P = V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \underbrace{\int d^dx\left\{\frac{K}{2}\left[(\nabla\varphi_\ell)^2 + |\nabla\varphi_t|^2\right] + \frac{t}{2}\varphi_\ell^2 + 6\bar{m}^2u\,\varphi_\ell^2 + \frac{t}{2}|\varphi_t|^2 + 2\bar{m}^2u|\varphi_t|^2\right\}}_{\text{Fluctuations in the quadratic approx}}$$

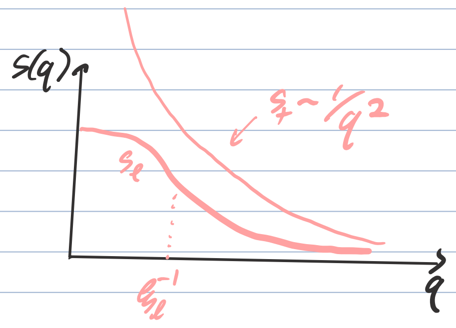
$$= V\Psi[\bar{m}] + K\int d^dx\left\{(\nabla\varphi_\ell)^2 + \xi_\ell^{-2}\varphi_\ell^2 + (\nabla\varphi_t)^2 + \xi_t^{-2}|\varphi_t|^2\right\}$$

$$\frac{K}{\xi_\ell^2} = \begin{cases} t & t>0 \\ -2t & t<0 \end{cases}$$

$$\frac{K}{\xi_t^2} = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases} \leftarrow \text{AKA no mass term}$$

$$\langle \varphi_{\alpha,q}\,\varphi_{\beta q'}\rangle = \frac{\delta_{\alpha\beta}\,\delta_{q,-q'}}{K(q^2 + \xi_\alpha^{-2})} \leftarrow \text{Lorentzian}$$





$\xi = \xi_\ell = \xi_t$

$$\langle \varphi_\alpha(x) \rangle = \langle m_\alpha(x) - \bar{m}_\alpha \rangle$$

$$G^c_{\alpha\beta} = \langle (m_\alpha(x) - \bar{m}_\alpha)(m_\beta(x') - \bar{m}_\beta) \rangle$$

$$= \langle \varphi_\alpha(x)\, \varphi_\beta(x') \rangle$$

$$= \frac{1}{V} \sum_{q,q'} e^{iq\cdot x + iq'\cdot x'} \langle \varphi_{\alpha q}\, \varphi_{\beta q'} \rangle$$

$$= \frac{\delta_{\alpha\beta}}{V} \sum_q \frac{e^{iq(x-x')}}{K(q^2 + \xi_\alpha^{-2})}$$

$$= \frac{\delta_{\alpha\beta}}{K} \underbrace{I_d(x-x', \xi_\alpha)}$$

$$- \int d^dq \, \frac{e^{iqx}}{q^2 + \xi^{-2}} \quad \Big\} \text{ Bessel}$$

$$\nabla I_d(x) = \int d^dq \, \frac{q^2}{q^2 + \xi^{-2}} e^{iqx} = \delta^d(x) + \frac{1}{\xi^2} I_d(x)$$

⇒ in spherical coords

$$\frac{d^2}{dr^2} I(r) + \frac{d-1}{r} \frac{dI_d}{dr} = \frac{I_d}{\xi^2} + \delta^d(x)$$

Try $I = \dfrac{e^{-r/\xi}}{r^p}$

$$\Rightarrow \begin{cases} I_d' = -\left(\dfrac{p}{r} + \dfrac{1}{\xi}\right) I_d \\ I_d'' = \left(\dfrac{p(p+1)}{r^2} + \dfrac{2p}{\xi r} + \dfrac{1}{\xi^2}\right) I_d \end{cases}$$

Choosing $\tilde{\xi} = \xi$

For $x \neq 0$

$$\Rightarrow \quad \frac{p(p+1)}{r^2} + \frac{2p}{r\tilde{\xi}} - \frac{p(d-1)}{r^2} - \frac{d-1}{r\xi} = 0$$

1) For $r \ll \xi$

$$p(p+1) = p(d-1) \Rightarrow \quad p = d-2 \Leftarrow \text{Coulomb}$$

$$\Rightarrow \quad I_d \propto \frac{1}{r^{d-2}}$$

2) For $r \gg \xi$

$$p = \frac{d-1}{2} \Rightarrow I_d \propto \frac{e^{-r/\xi}}{r^{\frac{d-1}{2}}} \times \frac{1}{\xi^{d/2 - 3/2}}$$

For correct dimensions

$$\xi_\ell = \frac{1}{\sqrt{K}} \times \begin{cases} \sqrt{t} \\ \sqrt{-2t} \end{cases} = \xi_0 B_\pm |t|^{-1/2}$$

$$\underbrace{\begin{matrix} \nu_+ = \nu_- = 1/2 \\ \frac{B_+}{B_-} = 2 \end{matrix}}_{\text{universal}} \qquad \underbrace{\xi_0 = \frac{1}{\sqrt{K}}}_{\text{not}}$$

At T_c $\quad \xi \to \infty \Rightarrow G_c \sim \frac{1}{r^{d-2+\eta}}$ $\qquad \eta = 0$

For $t > 0$:

$$\chi_\ell = \int d^dx \, G_\ell^c(r) \propto \int_0^{\xi_\ell} \frac{d^dx}{x^{d-2}} \propto \xi_\ell^2 \simeq A_\pm t^{-1}$$

For $t < 0$

$$\chi_t = \int d^dx \, G_t^c \propto \int_0^L \frac{d^dx}{x^{d-2}} \propto L^2$$

3.3 Lower Critical Dimension

For superfluid assume $|\psi|$ is uniform:

$$P[\theta(x)] \propto \exp\left[-\frac{\bar{K}}{2}\int d^dx\,(\nabla\theta)^2\right]$$

$$= \prod_q \exp\left[-\frac{\bar{K}}{2}|\theta_q|^2\right]$$

↑ each θ_q is indep Gaussian with $\langle\theta_q\,\theta_{q'}\rangle = \frac{\delta_{q,-q'}}{\bar{K}q^2}$

$$\langle\theta(x)\,\theta(x')\rangle = \frac{1}{V}\sum_{q,q'} e^{iqx+iq'x'}\langle\theta_q\,\theta_{q'}\rangle$$

$$= \frac{1}{V}\sum_q \frac{e^{iq(x-x')}}{Kq^2}$$

$$= \int d^dq\,\frac{e^{iq(x-x')}}{\bar{K}q^2} = -\frac{C_d(x-x')}{\bar{K}}$$

$$C_d(x) = -\int d^dq\,\frac{e^{ixq}}{q^2} \Rightarrow \nabla^2 C_d = \delta^d(x)$$

$$\Rightarrow \int d^dx\,\nabla^2 C_d = \oint dS\cdot\nabla C_d$$

$$\Rightarrow C_d = \frac{1}{r^{d-2}\,(d-2)\cdot S_d} + c_0$$

$$C_d(r\to\infty) = \begin{cases} c_0 & d>2 \\ \frac{1}{r^{d-2}}\cdot(d-2)S_d & d<2 \\ \frac{\log r}{2\pi} & d=2 \end{cases}$$

← decay

$$\Rightarrow \quad \langle [\theta(x) - \theta(x')]^2 \rangle = 2\langle \theta(x)^2 \rangle - 2\langle \theta(x)\theta(x') \rangle$$

as $x \to 0$ this $\to 0$ $\Rightarrow = \frac{2}{\bar{K}}\left[\frac{(x-x')^{2-d} - a^{2-d}}{(2-d)S_d}\right]$

$a \sim$ lattice spacing

$$\langle \psi(x)\psi(0) \rangle = \bar{\psi}^2 \langle e^{i[\theta(x)-\theta(0)]} \rangle$$

$$= \bar{\psi}^2 \exp\left[-\frac{1}{2}\langle [\theta(x)-\theta(0)]^2 \rangle\right]$$

$$= \bar{\psi}^2 \exp\left[-\frac{x^{2-d} - a^{2-d}}{\bar{K}(2-d)S_d}\right]$$

as $x \to \infty$ this becomes $\begin{cases} \bar{\psi}^2 & d > 2 \\ 0 & d \leq 2 \end{cases}$

Coleman - Mermin - Wagner

3.6 Fluctuation Corrections to Saddle point

$$Z \approx \exp\left[-V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right)\right] \int D[\phi_\ell, \phi_t] \exp\left[-\frac{K}{2}\int d^dq\, (q^2 + \xi_\ell^{-2})\, \phi_\ell^2 - \frac{K}{2}\int d^dq\, (q^2 + \xi_t^{-2})\, \phi_t^2\right]$$

$$\Rightarrow \beta F = \frac{-\log Z}{V} = \frac{t\bar{m}^2}{2} + u\bar{m}^4 + \frac{1}{2}\int d^dq \left[\log K(q^2 + \underset{\uparrow \begin{cases} t \\ -2t \end{cases}}{\xi_\ell^{-2}}) + (n-1)\log K(q^2 + \underset{\uparrow \begin{cases} t \\ 0 \end{cases}}{\xi_t^{-2}})\right]$$

$$\Rightarrow \quad C_{sing} \propto \frac{\partial^2 \beta F}{\partial t^2} = \begin{cases} 0 + \frac{n}{2}\int \frac{d^dq}{(Kq^2 + t)^2} \\ -\frac{1}{8u} + 2\int \frac{d^dq}{(Kq^2 - 2t)^2} \end{cases}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{correction terms}}$$

A correction term looks like

$$C_F := \frac{1}{K^2}\int \frac{d^dq}{(q^2+\xi^{-2})^2} \sim \text{length}^{4-d}$$

For $d > 4$ this diverges and is dominated by a^{4-d}

For $d < 4$ ⇒ converges and is $\propto \xi^{4-d}$

$$\Rightarrow C_F = \frac{1}{K^2}\begin{cases} a^{4-d} & d>4 \leftarrow \text{constant (large) term} \\ \xi^{4-d} & d<4 \leftarrow \text{corrects } \propto \text{ to } \frac{4-d}{2} \end{cases}$$

⇒ C_F diverges

The divergence of C_F below $d=4$ implies the saddle point conclusions are not reliable

We'd also see fluctuations modify behavior in m etc e.g. changes β, γ, δ etc

3.7 Ginzburg Criterion

We saw in the saddle point approx $\xi \approx \xi_0 |t|^{-1/2}$

$\xi_0 = \sqrt{K}$ is a microscopic length scale

it can be fit experimentally from the $S(q)$ curves

For the liquid-gas transition $\xi_0 \sim (v_c)^{1/3}$ ← critical atomic vol

For superfluids $\xi_0 \sim \lambda(T_c)$ the thermal wavelength

These are ~ 1 to 10 Å < 1e-9 m

But for superconductors $\xi_0 \approx 10^3$ Å ↑ avg cooper-pair distance

Importance of fluctuations is <u>relative</u>

Compare $\Delta C_{\text{saddle point}} = \frac{1}{8u}$

to $C_F = K^{-2}\xi^{4-d} = \xi_0^{-d}\, t^{\frac{4-d}{2}}$

Fluctuations matter if

$$\xi_0^{-d}\, t^{\frac{d-4}{2}} \gg \Delta C_{sp}$$

$$\Rightarrow |t| \ll \underbrace{t_G \approx \left(\xi_0^{d}\,\Delta C_{sp}\right)^{\frac{2}{d-4}}}_{\text{Ginzburg}}$$

For $d < 4$ taking $t \to 0$ will eventually satisfy this

$\xi_0 \sim a$
and $\Delta C_{sp} \sim Nk_B$ is $O(1)$

$\Rightarrow t_G = \xi_0^{-6}$ in $d=3$ eg

if $\xi_0 \sim a$ then $t_G \sim 10^{-1}$ works

But if $\xi_0 \sim 10^{3}\, a$ then $t_G < 10^{-18}$

For <u>any</u> quantity fluctuations always matter
at $t < t_G(X) \simeq A(X)\, \xi_0^{\frac{2d}{d-4}}$

4 The Scaling Hypothesis

4.1 The homogenaity assumption

Goal: Because various thermodynamic quantities are related, the exponents must be. Let's find the minimum # of indep exponents

Under the saddle point approximation:

$$f(t,h) = \min_m \left[\frac{t}{2} m^2 + u m^4 - hm \right] = \begin{cases} -\frac{1}{16}\frac{t^2}{u} & h=0 \quad t<0 \\ -3\left(\frac{h}{4}\right)^{4/3} \frac{1}{u^{1/3}} & h\neq 0 \quad t=0 \end{cases}$$

$$\Rightarrow \quad f(t,h) = |t|^2 g_f\left(\frac{h}{|t|^\Delta}\right) \leftarrow \text{gap exponent}$$

$$g_f(0) = \frac{1}{u}$$

as $x \to \infty$ $\quad g_f(x) = x^{4/3} \qquad 2 - 4\Delta/3 = 0 \Rightarrow \Delta = 3/2$

Assumption of homogeneity:

Even after accounting for fluctuations, the singular part of the free energy retains its homogenous form

$$f_{sing}(t,h) = t^{2-\alpha} g_f\left(h/|t|^\Delta\right)$$

$$\Rightarrow \quad E_{sing} \sim \frac{\partial f}{\partial t} \sim (2-\alpha)\, t^{1-\alpha} g_f - \Delta h\, t^{1-\alpha-\Delta} g_f'$$

$$\sim |t|^{1-\alpha}\left((2-\alpha) g_f - \Delta \frac{h}{|t|^\Delta} g' \right)$$

$$= |t|^{1-\alpha} g_E\left(\frac{h}{|t|^\Delta}\right)$$

$$\Rightarrow \quad C_{sing} \sim -\frac{\partial f}{\partial t} \sim |t|^{-\alpha} g_c\left(\frac{h}{|t|^\Delta}\right)$$

We cannot postulate $C_\pm = |t|^{-\alpha_\pm} g_\pm\left(\frac{h}{|t|^{\Delta_\pm}}\right)$

because away from $h=0$ $t<0$ f is analytic
$\Rightarrow$ at $t=0$ h finite

$$C_{sing}(t \ll h^\Delta) = A(h) + t B(h) + O(t^2)$$

$$C_{\pm} = |t|^{-\alpha_{\pm}}\left[A_{\pm}\left(\frac{h}{t^{\Delta_{\pm}}}\right)^{p_{\pm}} + B_{\pm}\left(\frac{h}{t^{\Delta_{\pm}}}\right)^{q_{\pm}}\right]$$

Matching yields $-p_{\pm}\Delta_{\pm} - \alpha_{\pm} = 0$

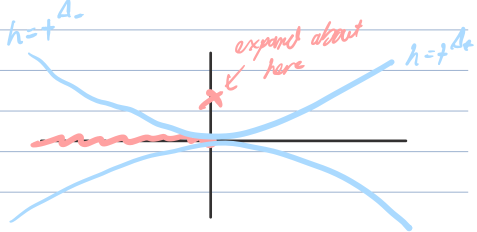
$$-q_{\pm}\Delta_{\pm} - \alpha_{\pm} = 1$$

$$C_{\pm}(t^{\max(\Delta_+,\Delta_-)} \ll h) = A_{\pm}\, h^{-\alpha_{\pm}/\Delta_{\pm}} + B_{\pm}\, h^{-(1+\alpha_{\pm})/\Delta_{\pm}}\, |t|$$

Continuity at $t=0 \Rightarrow \frac{\alpha_+}{\Delta_+} = \frac{\alpha_-}{\Delta_-} \qquad \frac{1+\alpha_+}{\Delta_+} = \frac{1+\alpha_-}{\Delta_-}$

$$\Rightarrow \underset{\alpha}{\underset{\shortparallel}{\alpha_+ = \alpha_-}} \qquad \underset{\Delta}{\underset{\shortparallel}{\Delta_+ = \Delta_-}}$$

$$A_+ = A_- \qquad B_+ = -B_-$$



$$m(t,h) \sim \frac{\partial f}{\partial h} = t^{2-\alpha-\Delta} g_m\left(\frac{h}{|t|^{\Delta}}\right)$$

$$m(t,0) \sim t^{2-\alpha-\Delta}$$

$$\Rightarrow \beta = 2-\alpha-\Delta$$

$$m(0,h) = h^{p} \qquad \Delta p = 2-\alpha-\Delta$$

$$= h^{\frac{2-\alpha-\Delta}{\Delta}} = h^{\frac{\beta}{\Delta}}$$

$$\Rightarrow \delta = \frac{\Delta}{\beta}$$

$$\chi(t,h) \sim \frac{\partial m}{\partial h} = t^{2-\alpha-2\Delta} g_{\chi}\left(\frac{h}{t^{\Delta}}\right)$$

$$\Rightarrow \gamma = 2\Delta + \alpha - 2$$

1) Singular parts of all $Q(t,h)$ are homogenous
same exponents above & below

2) Same gap exponent Δ

3) All exponents follow only from α, Δ

4) Exponent identities

i) $\delta - 1 = \gamma/\beta$ (Widom)

ii) $\alpha + 2\beta + \gamma = 2$ (Rushbrooke)

4.2 Divergence of ξ

Homogeneity says nothing about correlation functions

Need 2 new assumptions (Generalized homogeneity)

1) $\xi(t,h) = |t|^{-\nu} g(h/|t|^{\Delta})$ $\left(\Rightarrow \xi(0,h) \sim |h|^{-\nu/\Delta} \quad \nu_h = \nu/\Delta\right)$

2) Near criticality, ξ is the most important length and is solely responsible for the singular behavior

$$\log Z = \left(\frac{L}{\xi}\right)^d \underbrace{g_s}_{\text{nonsing}} + \cdots + \left(\frac{L}{a}\right)^d \underbrace{g_a}_{\text{nonsing}}$$

$$\Rightarrow f_{sing} \sim \frac{\log Z}{L^d} \sim \xi^{-d} \sim |t|^{d\nu} g_f\left(\frac{h}{|t|^{\Delta}}\right)$$

As a consequence of this

i) Homogeneity of f_{sing} comes naturally

ii) Additional relation

$$2 - \alpha = d\nu \quad \text{(Josephson)}$$

This is inconsistent w/ the saddle point solution $\alpha = 0$ $\nu = 1/2$ away from $d = 4$

Why does this breakdown for $d > 4$?

4.3 Critical Correlators

$$G_m^c := \langle m(x)\, m(0) \rangle_c \sim \frac{1}{|x|^{d-2+\eta}}$$

$$G_{\mathcal{H}}^c := \langle \mathcal{H}(x)\, \mathcal{H}(0) \rangle_c \sim \frac{1}{|x|^{d-2+\eta'}}$$

$$\chi \sim \int d^dx\, G^c_m(x) \sim \int_0^{\xi} \frac{d^dx}{|x|^{d-2+\eta}} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}$$

$$\Rightarrow \quad \gamma = \nu(2-\eta)$$

$$C \sim \int d^dx\, G^c_{\mathcal{E}}(x) \sim \xi^{2-\eta'} \sim |t|^{-\nu(2-\eta')}$$

$$\Rightarrow \quad \alpha = \nu(2-\eta')$$

by Josephson
↓

η, η' recover α, ν, γ

$$G_{critical}(\lambda x) = \lambda^p G_{critical}(x)$$ "self-similarity"

4.4 RG (Conceptual)

1) ξ is most important as you approach criticality
2) Fluctuations are self-similar up to scale ξ

This self-similarity is purely statistical

Idea (Kadanoff):

Gradually eliminate correlated d.o.f. until one is left with only simple uncorrelated d.o.f. at scale ξ

1) Coarse grain: Change $a \to ba$

$$m_i(x) = \frac{1}{b^d}\int_{\text{cell at } x} d^dx'\; m(x')$$

2) Rescale: $x_{new} = \frac{x_{old}}{b}$

3) Renormalize: The variance of the rescaled fluctuations is different. Introduce ζ

$$\vec{m}_{new}(x_{new}) = \frac{1}{\zeta b^d} \int_{\text{cell at } bx_{new}} d^d x' \, \vec{m}(x')$$

This is a mapping from one probability distribution to another

The insight of Kadanoff was that, since on length scales $\ll \xi$ the renormalized configs are statistically similar, they may be distributed according to a Hamiltonian $\beta \mathcal{H}_b$ that is also "close" to the original.

At $t = h = 0$ $\beta\mathcal{H}_b = \beta\mathcal{H}_1$

Kadanoff postulated that $\beta\mathcal{H}_b$ away from $t=h=0$ is described simply by t_{new}, h_{new}

$$\left.\begin{aligned} t_{new} &= t_b(t_{old}, h_{old}) \\ h_{new} &= h_b(t_{old}, h_{old}) \end{aligned}\right\} \text{must be analytic for } b \text{ close enough to } 1$$

$$t_b(t,h) = A(b)\,t + \cancel{B(b)\,h} + \dots \quad \text{vanishes by } \mathbb{Z}_2$$

$$h_b(t,h) = \cancel{C(b)}\,t + D(b)\,h + \dots$$

Because of the semigroup property, $A(b) = b^{y_t}$ $D(b) = b^{y_h}$

$$t' = b^{y_t}\, t + \dots$$
$$h' = b^{y_h}\, h + \dots$$

$\xi' = \xi / b$ ⇒ params move away from $(0,0)$ ⇒ $y_t, y_h > 0$

1) Free energy

$$Z = Z' \Rightarrow \log Z = \log Z'$$

$$\Rightarrow V f = V' f'$$

$$f = b^{-d} f'$$
$$= b^{-d} f(b^{y_t} t, b^{y_h} h)$$

let $b = t^{-1/y_t}$

$$\Rightarrow f = t^{d/y_t} f(1, h/t^{y_h/y_t})$$

$$\Rightarrow 2-\alpha = d/y_t$$ All other exponents follow!
$$\Delta = y_h/y_t$$

$$\alpha = 2 - d/y_t$$

$$\beta = \frac{d-y_h}{y_t}$$

$$\gamma = \frac{2y_h - d}{y_t} = 2\Delta-(2-\alpha) = \beta(\delta-1)$$

$$\delta = \frac{y_h}{d-y_h} = \frac{\Delta}{\beta}$$

2) Correlation length

$$\xi(t,h) = b\,\xi'$$
$$= b\,\xi(b^{y_t} t, b^{y_h} h)$$
$$= t^{-1/y_t} \xi(1, h/t^{y_h/y_t})$$

$$\Rightarrow \nu = 1/y_t$$

3) Magnetization

$$m = -\frac{1}{V} \frac{\partial \log Z}{\partial h} = -\frac{1}{b^d V'} \frac{\partial \log Z'(t', h')}{b^{-y_h} \partial h'}$$

$$= b^{y_h - d} m(b^{y_t} t, b^{y_h} h)$$

after $b = t^{-1/y_t}$

$$\Rightarrow \beta = \frac{y_h - d}{y_t} \qquad \Delta = \frac{y_h}{y_t}$$ as before

If $\int d^dx\, F \cdot X$ is in $\mathcal{H}$

$$y_x = y_F - d \qquad F' = b^{y_F} F$$

4.5 RG (Formal)

Main q: Why should $\mathcal{H}$, $\mathcal{H}'$ have the same form, with all effects absorbed into t', h'

1) Start w/ most general $\mathcal{H}$

$$\beta\mathcal{H} = \int d^dx \left[\frac{t}{2}m^2 + um^4 + vm^6 + \dots + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \dots\right]$$

2) Apply RG:

$$m'(x') = \frac{1}{\zeta b^d}\int_{\text{cell at } bx'} d^dx \; m(x)$$

3) New $\mathcal{H}$ has the same form, with all params different

$\Rightarrow$ Flow in parameter space induced by R_b

4) Fixed points of R_b have either $\xi = 0$ or $\xi = \infty$

$\uparrow$ $T = 0$ or $T = \infty$ indep vars at each site

$\uparrow$ critical point

5) Consider linearizing near a fixed point

Under RG the vector of params has

$$S^*_\alpha + \delta S'_\alpha = S^*_\alpha + (R^L_b)_{\alpha\beta}\,\delta S_b$$

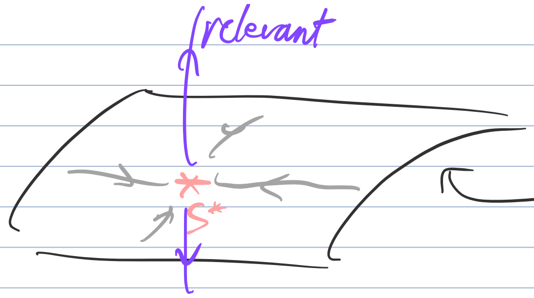
$$(R^L_b)_{\alpha\beta} = \left.\frac{\partial S'_\alpha}{\partial S_b}\right|_{S^*}$$

diagonalize:

evecs O_i w/ evals $\lambda(b)$. $\lambda(b) = b^{y_i}$ by semigroup property

Any $\mathcal{H}$ near S^* has the params: $S = S^* + \sum_i g_i O_i$

i) $y_i > 0 \Rightarrow g_i$ increases $\Rightarrow O_i$ relevant
ii) $y_i < 0 \Rightarrow g_i$ decreases $\Rightarrow O_i$ irrelevant
iii) $y_i = 0 \Rightarrow O_i$ marginal, need higher order



irrel op subspace, basin of attraction

$\xi(S^*) = \infty$

$$\xi(g_1, g_2, \dots) = b\,\xi(b^{y_1} g_1, b^{y_2} g_2, \dots)$$ microscopic details

⇒ For sufficiently large b, irrelevant couplings scale to 0

⇒ relevant determine all critical exponents

$$\xi(g_1, g_2, \dots) = g_1^{-1/y_1} f\left(\frac{g_2}{g_1^{y_2/y_1}}, \dots\right)$$

⇒ $\nu_1 = 1/y_1$

$\Delta_\alpha = y_\alpha / y_1$

People were nontheless unsure how to implement Kadanoff's ideas until Wilson showed how it could be done in the LG model

4.6 The Gaussian Model (direct solution)

$$Z = \int D\vec{m}(x) \exp\left\{-\int d^d x \underbrace{\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \dots + h\cdot m\right]}_{\text{up to } \mathcal{O}(m^2) \text{ only}}\right\}$$

Only defined for $t \geq 0$

$$m(q) = \int d^d x\, e^{iqx} m(x)$$

$$m(x) = \frac{1}{V}\sum_q e^{-iqx} m(q)$$

$$\Rightarrow \beta \mathcal{H} = \frac{1}{V}\sum_q \left(\frac{t}{2} + \frac{Kq^2}{2} + \frac{L}{2}q^4 + \dots\right)|m(q)|^2 - h \cdot m(q=0)$$

$$\Rightarrow Z = \prod_q V^{-n/2}\int dq \ \exp[-\beta\mathcal{H}]$$

Integrate $q=0$:

$$V^{n/2}\int_{-\infty}^{\infty} dm(q=0)\exp\left[-\frac{t}{2V}|m|^2 + h\cdot m\right] = \left(\frac{2\pi}{t}\right)^{n/2}\exp\left[\frac{Vh^2}{2t}\right]$$

For $q \neq 0$

$$\Rightarrow Z = \exp\left[\frac{Vh^2}{2t}\right]\prod_q\left(\frac{2\pi}{t+Kq^2+Lq^4+\dots}\right)^{n/2}$$

$$\Rightarrow f = -\frac{h^2}{2t} + \frac{n}{2}\int_{BZ} dq \ \log(t+Kq^2+Lq^4) + \text{const}$$

near BZ, log can be expanded in powers of t $\Rightarrow$ analytic

$\Rightarrow$ Focus on $q \approx 0$. WLOG BZ is sphere

$$\Rightarrow f_{sing} = \frac{n}{2}\frac{S_d}{(2\pi)^d}\int_0^{\Lambda} dq \ q^{d-1}\log(t+Kq^2+Lq^4) - \frac{h^2}{2t}$$

$$q = \sqrt{\frac{t}{K}}\ x$$

$$= \frac{n}{2}\frac{S_d}{(2\pi)^d}\left(\frac{t}{K}\right)^{d/2}\int_0^{\Lambda\sqrt{\frac{t}{K}}} dx \ x^{d-1}\left[\log t + \log\left(1 + x^2 + \overbrace{\frac{Lt}{K^2}x^4}^{\text{irred}} + \dots\right)\right] - \frac{h^2}{2t}$$

$$\Rightarrow f_{sing} = t^{d/2}\left[A + \frac{h^2}{2t^{1+d/2}}\right] + t^{d/2}\log t$$

$\uparrow$ less sing?

$$\Rightarrow \alpha = 2 - d/2 \qquad \beta \ \text{undef}$$

$$\Delta = 1/2 + d/4 \qquad \gamma = 1$$

4.7 Gaussian model (RG)

1) Coarse grain

$$\vec{m} = \vec{\sigma}(q^>) \oplus \vec{m}(q^<)$$

$$Z = \int Dm(q^<)\, D\sigma(q^>)\, e^{-\beta\mathcal{H}}$$

modes are decoupled in $\mathcal{H}$

$$\Rightarrow Z \sim \exp\left[-\frac{n}{2}V\int_{\Lambda/b}^{\Lambda} d^dq \,\log(t+Kq^2+\dots)\right.$$
$$\times \int Dm(q^<) \exp\left[-\int_0^{\Lambda/b} d^dq \left(\frac{t+Kq^2+\dots}{2}\right)|m(q)|^2 + h\cdot m(0)\right]$$

2) Rescale:

$$e^{-V\delta f_b} \int D\vec{m} \exp\left[-b^{-d}z^2\int_0^{\Lambda} dq' \left(\frac{t+Kq^2+\dots}{2}\right)|m(q)|^2 + z\,h\cdot m(0)\right]$$

z for $m(q)$ is not ζ for $m(x)$

$$\Rightarrow \quad \begin{aligned} t' &= z^2 b^{-d} t \\ h' &= z h \\ K' &= z^2 b^{-d-2} K \\ L' &= z^2 b^{-d-4} L \end{aligned}$$

$z = b^{1+d/2}$ $\Rightarrow$ t, h rel $\Rightarrow$ $y_t = 2$, $y_h = 1+d/2$

$\uparrow$ $\quad$ K marg

by ensuring $\mathcal{H}^*$ inv $\quad$ L, etc irrel

$\Rightarrow \nu = 1/y_t = 1/2$

$\Delta = \frac{y_h}{y_t} = \frac{1}{2} + \frac{d}{4}$

$\alpha = 2 - d\nu = 2 - \frac{d}{2}$

$$\beta\mathcal{H}^* = \frac{K}{2}\int d^dx\,(\nabla m)^2$$

agrees!

$x' = x/b$

$m' = m/\zeta$ $\Rightarrow$ $K' = b^{d-2}\zeta^2 K$ $\Rightarrow$ $\zeta = b^{1-d/2}$

$$\beta H^* + u_n \int m^n \to \beta H^* + \underbrace{u_n b^d \zeta^n}_{u_n'} \int (m')^n$$

$$\Rightarrow u_n' = b^d \zeta^n = b^{\underbrace{d+n-dn/2}_{y_n}}$$

Most ops are irrel for $d>2$

5 Perturbative RG

5.1 Expectation Values in the Gaussian Model

$$\beta H = \beta H_0 + U = \int d^dx \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 + \dots \right] + u \int d^dx \, m^4 + \dots$$

mixes q-modes

sum over α, β implicit (Einsum)

$$U = u \int d^dq_1 \, d^dq_2 \, d^dq_3 \, m_\alpha(q_1) \, m_\alpha(q_2) \, m_\beta(q_3) \, m_\beta(-q_1-q_2-q_3) + \dots$$

$$\langle m_\alpha(q) \, m_\beta(q') \rangle_0 = \frac{\delta_{\alpha\beta} \, \delta_{q,-q'} \cdot V}{t + Kq^2 + Lq^4} \to \frac{\delta_{\alpha\beta} \, \delta(q+q')}{t + Kq^2 + Lq^4}$$

non interacting

$\Rightarrow \left\langle \prod_i m_i \right\rangle_0$ evaluated by Wick

5.2 Expectation Values in Perturbation Theory

$$\langle O \rangle = \frac{\int Dm \, O \, e^{-\beta H_0 - U}}{\int Dm \, e^{-\beta H_0 - U}} = \frac{Z_0 [\langle O_0 \rangle - \langle O U_0 \rangle + \dots]}{Z_0 [1 - \langle U_0 \rangle + \dots]}$$

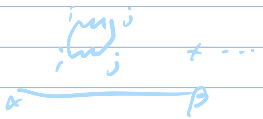
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle O U^n \rangle_0^c$$

$$\Rightarrow \langle m_\alpha(q)\, m_\beta(q')\rangle = \langle m_\alpha(q)\, m_\beta(q')\rangle_0$$

$5\cdot3\cdot1 = 15$ contractions

$$- u\int d^d q_{1,2,3}\Big[\langle m_\alpha(q) m_\beta(q')\, m_i(q_1)\, m_i(q_2)\, m_j(q_3)\, m_j(q_4)\rangle_0$$

$$- \langle m_\alpha m_\beta\rangle_0 \langle m_i^1\, m_i^2\, m_j^3\, m_j^4\rangle_0\Big]$$



3 bubbles

bubbles cancel

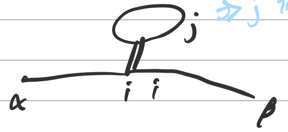
All variants of  For $n=1$

$$\Rightarrow -12\,u \times \frac{\delta_{\alpha\beta}\,\delta(q+q')}{(t+Kq^2)^2}\int_0^\Lambda \frac{d^dk}{t+Kk^2}$$

correctly defining $u \to \frac{u}{4!}$ gives $-\frac{u}{2} \times$ "

From twisting 

For generic n:

4x 

$\Rightarrow$ j free $= -4nu \times$ "

$\Rightarrow i=j=\alpha=\beta$ $= -8u \times$ "

in this case rescaling $u \to \frac{u}{8}$

$-u\left(\frac{1}{2}\cdot n + 1\right)$

twisting 

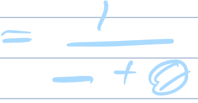
$$\Rightarrow \langle m_\alpha(q)\, m_\beta(q')\rangle = \frac{\delta_{\alpha\beta}\,\delta^d(q+q)}{t+Kq^2}\left[1 - \frac{4u(n+2)}{t+Kq^2}\int_0^\Lambda \frac{d^dk}{t+Kk^2} + O(u^2)\right]$$

5.4 Susceptibility

$$\langle m_\alpha(q)\, m_\beta(q')\rangle = \delta^d(q+q')\cdot S(q) \leftarrow S\text{-amplitude } \langle |m(q)|^2\rangle$$

$$\Rightarrow S(q) = \frac{1}{t+Kq^2}\left[1 - \frac{4u(n+2)}{t+Kq^2}\int_0^\Lambda \frac{d^dk}{t+Kk^2} + O(u^2)\right]$$

Resum! 



$$S(q)^{-1} = t + Kq^2 + 4u(n+2)\int_0^\Lambda \frac{d^dk}{t+Kk^2} + O(u^2)$$

$S(q) = \chi(q)$

$S(0) = \chi$

$$\Rightarrow \chi^{-1}(t) = t + 4u(n+2)\int_0^{\Lambda} \frac{d^dk}{t+Kk^2} + \cdots$$

$$\Rightarrow \chi^{-1}(0) = \frac{4u(n+2)}{K}\,\frac{\Lambda^{d-2}}{d-2}\,\frac{S_d}{(2\pi)^d} \neq 0$$

Isn't this huge? (not if Λ is small...)
to be done next

RG condition:

t_c is given by asking that $\chi^{-1}(t_c) = 0$

$$\Rightarrow t_c = -4u(n+2)\int \frac{d^dk}{t_c+Kk^2} \approx -\frac{4u(n+2)}{K}\,\frac{\Lambda^{d-2}}{d-2}\,\frac{S_d}{(2\pi)^d}$$

$\leftarrow$ infinite shift

Think of this as a mass shift!

I've basically just added and subtracted t_c

$$\underbrace{\chi^{-1}(t) - \chi^{-1}(t_c)}_{\text{How the perturbed } \chi \text{ diverges at } t_c} = t - t_c + 4u(n+2)\int d^dk \underbrace{\left[\frac{1}{t+Kk^2} - \frac{1}{t_c+Kk^2}\right]}_{\frac{t_c-t}{(t+Kk^2)(t_c+Kk^2)} \approx \frac{t_c-t}{Kk^2(t-t_c+Kk^2)} + O(u^2)}$$

$$= (t-t_c)\left[1 - \underbrace{\frac{4u(n+2)}{K^2}\int_0^{\Lambda} \frac{d^dk}{k^2\left(k^2+\frac{t-t_c}{K}\right)}}_{} + O(u^2)\right]$$

For $d>4$ dominated by $k \sim \Lambda$

For $2<d<4$ it is convergent

$\rightarrow$ Scales as $(\xi)^{4-d}$ $\quad \xi = \sqrt{\frac{K}{t-t_c}}$

$$= (t-t_c)\left[1 + \frac{4u(n+2)}{K^2}\, c \left(\frac{K}{t-t_c}\right)^{2-d/2} + O(u^2)\right]$$

$\uparrow$ diverges at $t \to t_c$, masking $\chi \sim t^{-1}$

u is not dimensionless

$\frac{u}{K^2}$ has units of length^{d-4}

$$X(t,u) = X_0(t)\left[1 + F\left(\frac{u}{K^2}\cdot a^{4-d}, \frac{u}{K^2}\xi^{4-d}\right)\right.$$

$\nearrow$ diverges for $d<4$

near T_c

⇒ Perturbation theory fails

5.5 Perturbative RG

Wilson showed how to combine perturbative & RG approaches

1) Coarse grain

$$Z = \int Dm D\sigma \exp\left\{-\int_0^\Lambda d^dq \left[\left(\frac{t+Kq^2}{2}\right)(|m|^2+|\sigma|^2) - \underset{\uparrow \text{ mixes}}{U(m,\sigma)}\right]\right\}$$

$$= \int Dm D\sigma \exp\left\{-\int_0^{\Lambda/b} d^dq \, \frac{t+Kq^2}{2}|m|^2\right\} \underbrace{\exp\left\{-\frac{nV}{2}\int_{\Lambda/b}^{\Lambda} d^dq \log(t+Kq^2)\right\}}_{Z_\sigma = \exp[-V\delta f_b^0]} \left\langle e^{-U[m,\sigma]}\right\rangle_\sigma$$

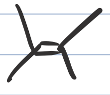
$$= \int D\tilde{m} \exp\left[-\beta \mathcal{H}[\tilde{m}]\right]$$

$$\langle O\rangle_\sigma := \int \frac{d\sigma}{Z_\sigma}\, O \exp\left\{-\int_{\Lambda/b}^{\Lambda} d^dq \left(\frac{t+Kq^2}{2}\right)|\sigma|^2\right\}$$

$$\Rightarrow \beta\mathcal{H} = V\delta f_b^0 + \int_0^{\Lambda/b} d^dq \, \frac{t+Kq^2}{2}|m|^2 - \log\left\langle e^{-U[m,\sigma]}\right\rangle_\sigma$$

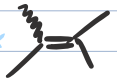
$$\log\langle e^{-U}\rangle_\sigma = \underbrace{-\langle U\rangle_\sigma^c}_{\text{1st order}} + \frac{1}{2}\langle U^2\rangle_\sigma^c + \cdots + \frac{(-1)^\ell}{\ell!}\langle U^\ell\rangle_\sigma^c$$

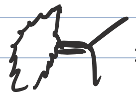
$$\langle U\rangle_\sigma = u\int d^dq_{1234} \underbrace{(m_1+\sigma_1)(m_2+\sigma_2)(m_3+\sigma_3)(m_4+\sigma_4)}_{\text{16 diagrams}} \delta^4$$

1) 1x  $= U[\tilde{m}]$

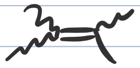
$\sim = \sigma$

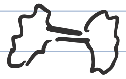
$— = m$

2) 4x  $= 0$

3) 2x  =  $= -\frac{4nu}{2}\int_0^{\Lambda/b} d^dq\,|m|^2 \underset{\swarrow \text{ shifts } t}{\int_{\Lambda/b}^{\Lambda}} \frac{d^dk}{t+Kk^2}$

4) 4x  =  $= -4u$ //

5) 4x  $= 0$

6) 1x  =  +  $= u\, V \delta f_b'$

$$\Rightarrow \beta \mathcal{H}[\tilde{m}] = V(\delta f_b^0 + u\, \delta f_b') + \int_0^{\Lambda/b} d^dq \left(\frac{\tilde{t} + Kq^2}{2} \right) |\tilde{m}|^2 + u \int_0^{\Lambda/b} d^dq_{1,2,3,4}\, \delta^d\, \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \tilde{m}_4$$

$$\tilde{t} = t - 4u(n+2) \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{t + Kk^2}$$

$$\Rightarrow \tilde{K} = K \qquad \tilde{u} = u$$

2) Rescale $q = b^{-1} q'$

3) Renormalize $\tilde{m} = z m'$

$$= V(\delta f_b^0 + u\, \delta f_b') + b^{-d} z^2 \int_0^{\Lambda/b} d^dq' \left(\frac{\tilde{t} + Kb^{-2} q'^2}{2} \right) |m'|^2 + b^{-3d} z^4 u \int_0^{\Lambda/b} d^dq_{1,2,3,4}\, \delta^d\, \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \tilde{m}_4$$

$$t' = b^{-d} z^2 \tilde{t}$$

$$K' = b^{-d-2} z^2 K$$

$$u' = b^{-3d} z^4 u$$

Fixed point at $t = u = 0$
provided
$z = b^{1+d/2}$

$$\Rightarrow \left.\begin{aligned} t' &= b^2 \left[t + 4u(n+2) \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{t + Kk^2} \right] \\ u' &= b^{4-d} u \\ K' &= K \end{aligned}\right\} + O(u^2)$$

$b = e^{\ell}$. In terms of $\delta\ell$:

$$\frac{dt}{d\ell} = 2t + \frac{4u(n+2)}{t+K\Lambda^2}\frac{S_d}{(2\pi)^d}\Lambda^d$$

$$\frac{du}{d\ell} = (4-d)u$$

$$\Rightarrow \quad u = u_0\, b^{4-d}$$

Near $t = u = 0$

$$\frac{d}{dt}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix} = \begin{pmatrix} 2 & \frac{4(n+2)}{K}\frac{S_d}{(2\pi)^d}\Lambda^{d-2} \\ 0 & 4-d \end{pmatrix}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix}$$

Evals are still 2, $4-d$

$u=0$, δt ↗ (for 2); ↑ (for $4-d$) $t = -\frac{4u(n+2)}{K}\frac{S_d}{(2\pi)^d}\Lambda^{d-2}$, increase u

We get no other fixed point.

However, since series is alternating in u, anticipate:

$$\frac{dt}{d\ell} = 2t + \frac{4u(n+2)\,K_d\Lambda^d}{t+K\Lambda^2} - Au^2$$

↑ don't care about this term

$$\frac{du}{d\ell} = \underbrace{(4-d)u - Bu^2}_{\text{new F.P. @ } u \sim \frac{4-d}{B}}$$

Take $\varepsilon = 4-d$ small!

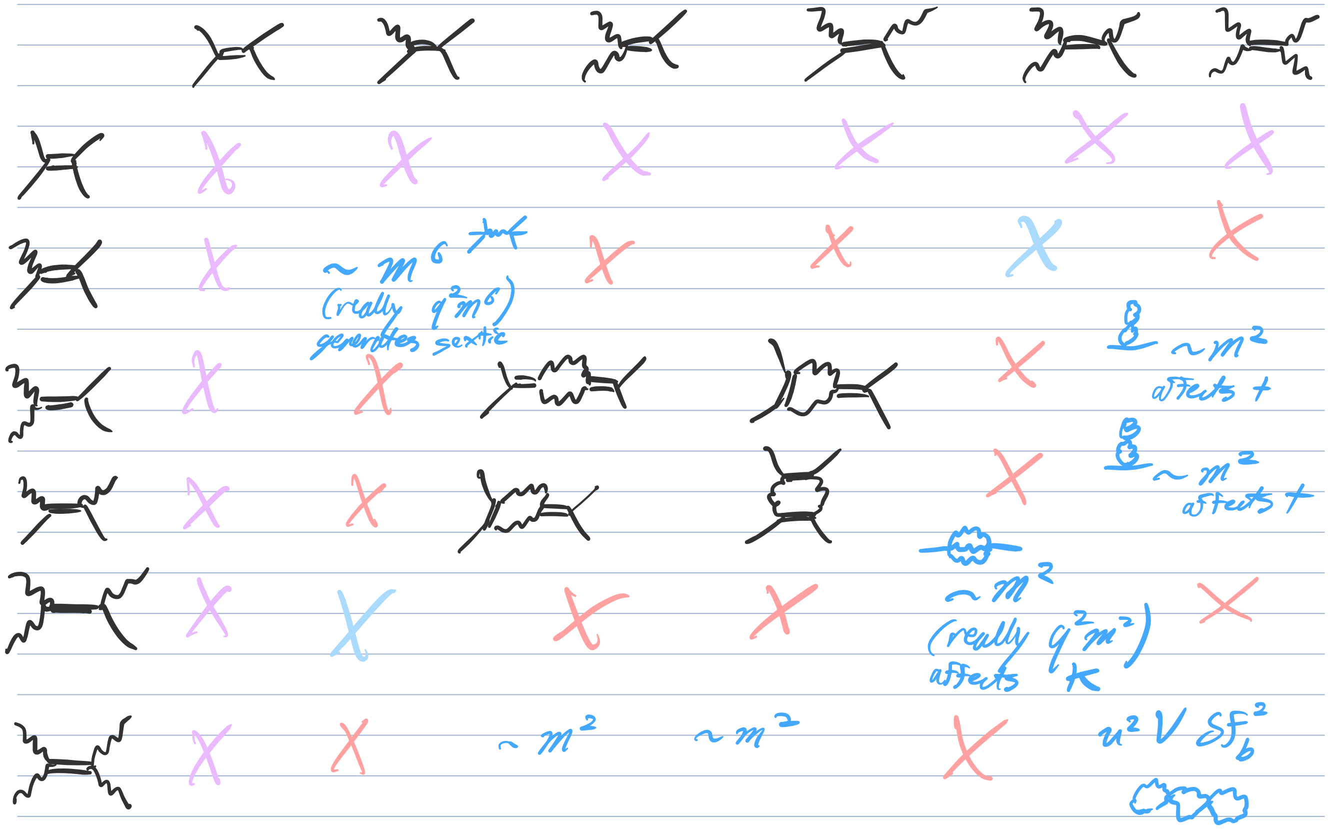
Wilson's ε-expansion

5.6 Perturbative RG @ 2nd order

$$\Rightarrow \beta\mathcal{H} = V\delta f_b^0 + \int_0^{\Lambda/b} d^dq\, \frac{t+Kq^2}{2}|m|^2 - \log\left\langle e^{-U[m,\sigma]}\right\rangle_\sigma$$

$$\log \langle e^{-U} \rangle_\sigma = -\langle U \rangle_\sigma^c + \underbrace{\tfrac{1}{2} \langle U^2 \rangle_\sigma^c}_{\text{this term}} + \dots + \frac{(-1)^\ell}{\ell!} \langle U^\ell \rangle_\sigma^c$$

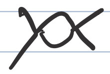
Before, we had 6 types of vertices. Now 36

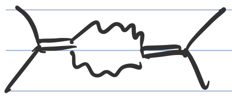


X = disconnected

X = parity

X = momentum (only one $\overset{m}{\sigma}$ leg the rest $\overset{\sigma}{m}$ can't conserve p by defn)

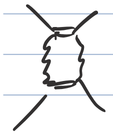
 variants run u



$$: \; = \frac{(8u)^2}{2 \cdot 2^3} \int_0^{\Lambda/b} d^d q_{1,2,3,4} \int_{\Lambda/b}^{\Lambda} d^d k_{1,2,3,4} \frac{\delta_{q_{12} k_{12}} \delta_{k_{12} q_{34}}}{(t + K k_1^2)(t + K k_2^2)} \times n \times \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \tilde{m}_4$$

$$= 4nu^2 \int_0^{\Lambda/b} \bar{d}^d q_{1234}\, \delta_{1234}\, \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \tilde{m}_4 \int_{\Lambda/b}^{\Lambda} \frac{\bar{d}^d k}{(t+Kk^2)(t+K(q_{1+2}-k)^2)}$$

$\uparrow$ generate $m^2 \nabla^2 m^2$ $m^2(\nabla m)^2$ etc



$$\sim \frac{(8u)^2}{2} \int_{\Lambda/b}^{\Lambda} \frac{\bar{d}^d k}{(t+Kk^2)^2} + \text{irrel}$$

$$\Rightarrow \tilde{K} = K - u^2 A''(0)$$

$$\tilde{t} = t + 4(n+2)u \int_{\Lambda/b}^{\Lambda} \frac{\bar{d}^d k}{t+Kk^2} - u^2 A(0)$$

$$\hat{u} = u - 4(n+8)u^2 \int_{\Lambda/b}^{\Lambda} \frac{\bar{d}^d k}{(t+Kk^2)^2}$$

$+O(u^3)$

$q = b^{-1} q'$

$$\Rightarrow K' = b^{-d-2} z^2 \tilde{K}$$

$$t' = b^{-d} z^2 \tilde{t}$$

$$u' = b^{-3d} z^4 \hat{u}$$

z is chosen so $K' = K$

$$\Rightarrow z^2 = \frac{b^{d+2}}{1-u^2 A''(0)} = b^{d+2}(1+O(u^2)) = b^{d+2+O(\epsilon^2)}$$

$\Rightarrow z = b^{1+d/2}$ to $O(\epsilon)$

$$\Rightarrow \frac{dt}{dl} = 2t + \frac{4u(n+2)}{t+K\Lambda^2} \frac{S_d}{(2\pi)^d} \Lambda^d - u^2 A(0)$$

$$\frac{du}{dl} = (4-d)u - \frac{4(n+8)}{(t+K\Lambda^2)^2} \frac{S_d}{(2\pi)^d} \Lambda^d u^2$$

Two F.P.s now: 1) $t^* = u^* = 0$

2) $$u^* = \frac{(t+K\Lambda^2)^2 \epsilon}{4(n+8)\, K_d \Lambda^d} = \frac{K^2}{4(n+8)K_4}\epsilon + O(\epsilon^2)$$

$$t^* = -\frac{2u^*(n+2) K_d \Lambda^d}{t^* + K\Lambda^2} = -\frac{n+2}{2(n+8)} K\Lambda^2 \epsilon + O(\epsilon^2)$$

Linearizing:

$$\frac{d}{dl}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} 2-\frac{4(n+2)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^* - A' u^{*2} & \frac{4(n+2)}{t^*+K\Lambda^2}K_d\Lambda^d - 2Au^* \\ \frac{8(n+8)}{(t+K\Lambda^2)^3}K_d\Lambda^d u^{*2} & 4-d-\frac{8(n+8)}{(t_*+K\Lambda^2)^2}K_d\Lambda^d u^* \end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}$$

At $t=u=0$

$$\rightarrow \begin{pmatrix} 2 & \frac{4(n+2)K_d\Lambda^{d-2}}{K} \\ 0 & 4-d \end{pmatrix}$$

At new FP:

$$\begin{pmatrix} 2-\frac{n+2}{n+8}\epsilon & \cdots \\ O(\epsilon^2) & \underbrace{\epsilon - \frac{8(n+8)K_4}{K^2}\frac{K^2\epsilon}{4(n+8)K_4}}_{-\epsilon\ \Rightarrow \text{irrel}} \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} y_t &= 2-\frac{n+2}{n+8}\epsilon + O(\epsilon^2)\\ y_u &= -\epsilon + O(\epsilon^2)\end{aligned}\right\} K,\ \Lambda\ \text{indep}\ \ddot\smile$$

$$\xi \sim (\delta t)^{-\nu} \qquad \nu = \frac{1}{y_t} = \frac{1}{2} + \frac{1}{4}\frac{n+2}{n+8}\epsilon + O(\epsilon^2)$$

$$f \sim (\delta t)^{2-\alpha} \qquad \alpha = 2-d\nu$$

$$= 2 - \frac{4-\epsilon}{2}\left[1+\frac{1}{2}\frac{n+2}{n+8}\epsilon\right]$$

$$= -\frac{n+2}{n+8}\epsilon + \frac{1}{2}\epsilon$$

$$= \frac{n+8-2n-4}{2(n+8)} = \frac{4-n}{2(n+8)}$$

Adding $-\vec{h}\cdot\vec{m}(q=0)$ to $\mathcal{H}$:

$$h' = zh = b^{1+d/2} \Rightarrow y_h = 1+\frac{d}{2} = 3-\frac{\epsilon}{2}+O(\epsilon^2)$$

$$\beta = \frac{d-y_h}{y_t} = \left(1-\frac{\epsilon}{2}\right)\left(\frac{1}{2}+\frac{1}{4}\frac{n+2}{n+8}\epsilon\right)$$

$$= \frac{1}{2}-\frac{\epsilon}{4}+\frac{1}{4}\frac{n+2}{n+8}\epsilon$$

$$= \frac{1}{2}-\frac{1}{2}\frac{3}{n+8}\epsilon$$

$$\gamma = \frac{2y_h-d}{y_t} = 2\left(\frac{1}{2}+\frac{1}{4}\frac{n+2}{n+8}\epsilon\right) = 1+\frac{1}{2}\frac{n+2}{n+8}\epsilon+O(\epsilon^2)$$

5.8 Irrelevance of other interactions

$$\beta\mathcal{H}^* = \frac{K}{2}\int_\Lambda d^dx\left[(\nabla m)^2-\frac{n+2}{n+8}\Lambda^2\epsilon\, m^2+\frac{\epsilon\Lambda^{-\epsilon}}{2(n+8)}\frac{K}{K_4}m^4\right]$$

explicitly depends on cutoff Λ

Higher order terms (eg $\propto m^6$ etc) were generated by coarse graining

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$$

Gaussian

all else: um^4 $vm^2(\nabla m)^2$

$u_6 m^6$ $u_8 m^8$

$x = bx'$ $\qquad q = b^{-1}q'$

$m(x) = \zeta m'(x')$ $\qquad m(q) = z\, m'(q')$

$t \rightarrow b^d \zeta^2 \tilde{t} = b^2 \tilde{t}$

$K \rightarrow b^{d-2} \zeta^2 \tilde{K} = K$

$L \rightarrow b^{d-4} \zeta^2 \tilde{L} = b^{-2} \tilde{L}$

$\vdots$

$u \rightarrow b^d \zeta^4 \tilde{u} = b^{4-d} \tilde{u}$

$v \rightarrow b^{d-2} \zeta^4 \tilde{v} = b^{2-d} \tilde{v}$

$\vdots$

$u_6 \rightarrow b^d \zeta^6 \tilde{u}_6 = b^{6-2d} \tilde{u}_6$

$u_8 \rightarrow b^d \zeta^8 \tilde{u}_8 = b^{8-3d} \tilde{u}_8$

Choosing $\zeta^2 = b^{2-d} \dfrac{K}{\tilde{K}} = b^{2-d}[1 + O(u; uv, \cdots)]$

$\Rightarrow K' = K$

At $t = u = L = \cdots = 0$

$y_t^0 = 2 \quad y_u = \epsilon$

all else are < 0 as $\epsilon \rightarrow$

Similar for other F.P.

only u gets corrected to be irrel